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Quantum deformations of the Heisenberg equations of motion

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Abstract. We consider various formulations of what may be called q-quantum mechanics. One involves replacing the commutator that appears in the Heisenberg equations of motion by the 'quommutator': $[A, B]_q = qAB - (1/q)BA$. We formally integrate the equations of motion, study some simple examples, and discuss various apparent difficulties such as lack of conservation of energy, lack of unitarity, and the unboundedness of the norm. We also consider another formulation in which the quantum bracket of the time derivative of an operator is used instead of just the time derivative, but the commutator is unchanged, in the Heisenberg equations of motion. This nonlinear treatment, while preserving conservation of energy, has its own set of difficulties due to the nonlinearities. We also suggest that the quantum parameter may function as a kind of regulator for the time evolution of the system.

1. Introduction

The notion of a quantum group [1] or, more properly, a quantized universal enveloping algebra, has recently received considerable attention in the physics literature in connection with conformal field theories, topological field theories, and exactly solvable lattice models, in all of which the Yang–Baxter equation plays a central role [2, 3]. In much of this work the role of quantum groups, while crucial in the analysis of these systems, has not been that of a direct symmetry.

However, in the work of Pasquier and Saleur [3] the generators of the quantum Lie algebra $\mathrm{su}(2)_q$ actually commute with the Hamiltonian of the one-dimensional spin chain they investigated, and so $\mathrm{su}(2)_q$ is a direct symmetry of the system. Yet in the cases when the deforming parameter q is not a root of unity, it has been pointed out [4] that, by making use of the deforming mappings constructed by Curtright and Zachos [5], these Hamiltonians are also necessarily invariant under ordinary su(2). So, at least in these $q^n \neq 1$ cases, it seems that these spin chain models do not exhibit any really new physics. This is confirmed by the identical representation content of the deformed algebra and its classical parent when $q^n \neq 1$. There are, however, striking and significant differences when q is a root of unity.

In connection with the study of quantum groups, there have been various investigations of deformed harmonic oscillator algebras and thereby deformed Heisenberg algebras [6]. One can regard this as a modification of one of the basic postulate relations of quantum mechanics. This, and all the above, raises the question of whether it may be possible to use the idea of a quantum deformation in a yet more central way, directly in the dynamics and structure of quantum mechanics itself, so that one considers a

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quantum-deformed quantum mechanics, or more compactly (and with less emphasis on the apparent pleonasm due to the misnominous *quantum* in quantum groups) a q-quantum mechanics. This is guaranteed to have immediate physical consequences; whether these consequences are desirable or even tolerable will be discussed at the end of this article. At the very best, one learns from such an investigation what is really essential for a mechanics to make sense, and what is only apparently so.

One of the definitions of quantum SU(2) is to modify the usual Lie algebra (here J_0 is twice its conventional value in quantum mechanics)

$$[J_0, J_\pm] = \pm 2J_\pm \tag{1a}$$

$$[J_+, J_-] = J_0 \tag{1b}$$

by setting

$$[J_0, J_\pm] = \pm 2J_\pm \tag{2a}$$

and

$$[J_+, J_-] = [J_0]_q \tag{2b}$$

where, for any x, $[x]_q = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$. As $q \to 1$, the quantum algebra reduces to the usual one.

There is another way to introduce the quantum deformation [7], which involves modifying the left-hand side of the SU(2) commutation relations rather than the right-hand side. We define the 'quommutator' $[A, B]_q$ of two operators A and B:

$$[A, B]_q \equiv qAB - \frac{1}{q} BA. \tag{3}$$

Then an equivalent way of deforming SU(2) is to write

$$[J_0, J_\pm]_{s^2} = \pm J_\pm \qquad [J_+, J_-]_{1/s} = J_0. \tag{4}$$

In this paper we will explore both methods to deform the basic dynamical laws of quantum mechanics itself.

2. *Q*-quantum mechanics

Let $X(t)$ be an observable in the theory. In the first method of introducing the deformation, we postulate that its evolution be given by the modified Heisenberg equation of motion (this deformation was briefly considered (and rejected) in [8]):

$$\dot{X}(t) = \mathrm{i}[H(t), X(t)]_q. \tag{5}$$

Here $H(t)$ is the Hamiltonian for the system. It is easy to see that H cannot be consistently taken to be a constant, because we can apply equation (5) to H itself:

$$\begin{aligned} \dot{H}(t) &= \mathrm{i}[H(t), H(t)]_q \\ &= \mathrm{i}rH^2(t) \end{aligned} \tag{6}$$

where we have written

$$r = q - 1/q. \tag{7}$$

The solution of equation (6) for H is

$$H(t) = \frac{H_0}{1 - \mathrm{i}rH_0 t} \tag{8}$$

where H_0 is a constant operator, the value of H at $t=0$. Now that we know $H(t)$, we can solve equation (5) formally for $X(t)$:

$$X(t)=(1-\mathrm{i}rH_0t)^{-q/r}X(0)(1-\mathrm{i}rH_0t)^{1/qr}. \tag{9}$$

If $[H_0, X(0)]=0$, we can use the fact that

$$-\frac{q}{r}+\frac{1}{qr}=-1 \tag{10}$$

to write

$$X(t)=\frac{X(0)}{1-\mathrm{i}rH_0t} \tag{11}$$

of which equation (8) is a special case.

We note, also, that

$$\lim_{q\to 1}(1-\mathrm{i}rH_0t)^{-q/r}=\exp(\mathrm{i}H_0t) \tag{12a}$$

and

$$\lim_{q\to 1}(1-\mathrm{i}rH_0t)^{1/qr}=\exp(-\mathrm{i}H_0t) \tag{12b}$$

so that as $q\to 1$ we indeed recover the familiar time dependence of an observable in the Heisenberg picture.

The operators $S=(1-\mathrm{i}rH_0t)^{1/qr}$ and $S'=(1-\mathrm{i}rH_0t)^{-q/r}$ cannot be inverse to each other (unless $q=\pm 1$) since, as we have remarked above, $S'S=1/(1-\mathrm{i}rH_0t)$. If we choose q to lie on the unit circle,

$$q=\mathrm{e}^{\mathrm{i}\theta} \qquad r=2\mathrm{i}\sin\theta \tag{13}$$

then, provided that $1+2\sin\theta\, H_0t>0$,

$$S'=S^{\dagger} \tag{14}$$

so that Hermiticity, at least, is preserved under time evolution. (Note that if one treats q as a formal variable, as is usually done with quantum groups, and so one does not complex conjugate it when taking the Hermitian adjoint of an expression in which q appears, then one cannot preserve Hermiticity. However, we do not have the formal structure of a quantum group here, so how one treats q appears to be a matter of choice.) No matter what the value of $\sin\theta$ ($\neq 0$) and whatever the spectrum of H_0, there will exist ranges of t for which the above inequality is violated, so there will be in general no guarantee that even Hermiticity will be preserved for all times. Nevertheless, since this appears to be the best that one can do, we shall confine ourselves henceforth to the case $q=\mathrm{e}^{\mathrm{i}\theta}$. We shall see below (equations (49), (54) and (55)) that Hermiticity is at least preserved in the case of a quantum-deformed free particle.

The reader may be concerned that when $q=\mathrm{e}^{\mathrm{i}\theta}$ and $r=2\mathrm{i}\sin\theta$, the factor $1-\mathrm{i}rH_0t=1+2\sin\theta\, H_0t$ can vanish, leading to singularities in expressions like, for example, equations (8) and (9). We shall return to this point after we discuss the time evolution of states.

Given the Heisenberg picture relation

$$X_{\mathrm{H}}(t)=S'(t)X_{\mathrm{H}}(0)S(t) \tag{15}$$

we are led in the usual way to the Schrödinger picture, in which the operators remain constant in time,

$$X_S(t) = X(0) = X_H(0) \tag{16}$$

but the states evolve,

$$|\Psi(t)\rangle = S(t)|\Psi(0)\rangle \tag{17}$$

and likewise

$$\langle\Psi(t)| = \langle\Psi(0)|S'(t). \tag{18}$$

We define the Hamiltonian in the Schrödinger picture, $\hat{H}_S$, as the solution to the equation

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}_S|\Psi\rangle. \tag{19}$$

We denote this operator with a hat because it will turn out not to be what one gets simply from equation (8). This will be discussed further below.

From equation (19) we deduce

$$\mathrm{i}\frac{\partial}{\partial t}S(t) = \hat{H}_S S \tag{20}$$

i.e. $\hat{H}_S = \mathrm{i}\dot{S}S^{-1} = (\mathrm{e}^{-\mathrm{i}\theta})H(t)$ with $H(t)$ given by equation (8). Note that $\hat{H}_S^\dagger = -\mathrm{i}S'^{-1}\dot{S}'$, in conformity with equation (18).

Perhaps it is worthwhile at this point to recapitulate the situation regarding the various Hamiltonians of the system.

We began with the definition (equation (5)) of $H(t)$ which governs time evolution via the modified Heisenberg equation of motion. Explicitly,

$$H(t) = \frac{H_0}{1+2(\sin\theta)H_0 t}. \tag{21}$$

In the Schrödinger representation we find, simply, that $H = H_0$. Note that both H and H_0 are Hermitian.

However, there is another operator, $\hat{H}$, defined as governing the time evolution of states in the Schrödinger picture:

$$\hat{H}(t) = \mathrm{e}^{-\mathrm{i}\theta}H(t). \tag{22}$$

$\hat{H}(t)$ is unusual in two respects: (i) even though it is defined in the Schrödinger picture it is time-dependent (this must be viewed as explicit time dependence); (ii) $\hat{H}(t)$ is not Hermitian, but its lack of Hermiticity is of a particularly simple form—it is $\mathrm{e}^{-\mathrm{i}\theta}$ times a Hermitian operator, which happens to be $H(t)$.

If we transform $\hat{H}(t)$ to the Heisenberg picture, we find

$$\hat{H}_H(t) = \frac{\mathrm{e}^{-\mathrm{i}\theta}H_0}{[1+2(\sin\theta)H_0 t]^2}. \tag{23}$$

The reason $\hat{H}_H$ fails to obey equation (11), even though it commutes with H_0, is because of its explicit time dependence.

Next we shall discuss the potential problem associated with the factor $1+2(\sin\theta)H_0 t$ that occurs in the denominator of the time evolution operator. Let us examine the expectation value of an operator $\mathcal{O}(t)$ that obeys equation (9), and that is therefore time-independent,

$$\mathcal{O}_S(t)=\mathcal{O} \tag{24}$$

in the Schrödinger representation. We then have

$$\bar{\mathcal{O}}(t)=\frac{\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle}. \tag{25}$$

Now

$$S(t)=(1+2\sin\theta\, H_0 t)^{1/qr}=(1+2\sin\theta\, H_0 t)^{1/2-(\mathrm{i}/2)\cot\theta} \tag{26}$$

and

$$S'(t)=(1+2\sin\theta\, H_0 t)^{-q/r}=(1+2\sin\theta\, H_0 t)^{1/2+(\mathrm{i}/2)\cot\theta}. \tag{27}$$

Expanding the numerator and denominator in eigenstates of H_0,

$$H_0|n\rangle=\lambda_n|n\rangle \tag{28}$$

(for simplicity we use a discrete notation, whether or not the spectrum of H_0 has a continuous region), we obtain

$$\bar{\mathcal{O}}(t)=\frac{\Sigma_{n,m}\langle\Psi_0|n\rangle(1+2\sin\theta\,\lambda_n t)^{-(1/2)+(\mathrm{i}/2)\cot\theta}\langle n|\mathcal{O}|m\rangle(1+2\sin\theta\,\lambda_m t)^{-1/2-(\mathrm{i}/2)\cot\theta}\langle m|\Psi_0\rangle}{\Sigma_n\langle\Psi_0|n\rangle(1+2\sin\theta\,\lambda_n t)^{-1}\langle n|\Psi_0\rangle}. \tag{29}$$

When $t\approx t_k=-1/2\sin\theta\,\lambda_k$, provided $\langle k|\Psi_0\rangle\neq 0$, the dominant term in the numerator is that for which $n=m=k$; likewise, the dominant term in the denominator is the one with $n=k$. Hence

$$\bar{\mathcal{O}}(t_k)=\langle k|\mathcal{O}|k\rangle \tag{30}$$

which is finite in general. So even though the norm of $|\Psi\rangle$ is infinite for $t=t_k$, the expectation value of an operator remains finite.

Likewise, if the spectrum of H_0 is bounded away from zero, so that there exist times for which $|2\sin\theta\,\lambda_n t|\gg 1\ \forall n$, then $\bar{\mathcal{O}}(t)$ also has a well defined limit as $|t|\to\infty$,

$$\lim_{|t|\to\infty}\bar{\mathcal{O}}(t)=\frac{\Sigma_{n,m}\langle\Psi_0|n\rangle\lambda_n^{-(1/2)+(\mathrm{i}/2)\cot\theta}\langle n|\mathcal{O}|m\rangle\lambda_m^{-1/2-(\mathrm{i}/2)\cot\theta}\langle m|\Psi_0\rangle}{\Sigma_n|\langle n|\Psi_0\rangle|^2\lambda_n^{-1}} \tag{31}$$

notwithstanding the fact that $\langle\Psi|\Psi\rangle$ is tending to zero in this limit.

Observe that the eigenstates of H_0 deserve the name 'stationary states' inasmuch as the expectation value of an operator $\mathcal{O}$ in such a state $|k\rangle$ is given by $\langle k|\mathcal{O}|k\rangle$ independent of time.

To examine the dynamics further, we shall, first, briefly study the classical mechanical case with deformed Poisson brackets. Second, we shall return to the quantum-mechanical case and look at a couple of simple examples.

To deform classical mechanics, we introduce the q-Poisson bracket

$$\{A,B\}_q\equiv q\frac{\partial A}{\partial x}\frac{\partial B}{\partial p}-\frac{1}{q}\frac{\partial B}{\partial x}\frac{\partial A}{\partial p} \tag{32}$$

where, for simplicity, we have assumed that there is only one degree of freedom. Then we modify Hamilton's equations to

$$\dot{x}=\{x, H\}_q \qquad \dot{p}=\{p, H\}_q. \tag{33}$$

If H is a conventional Hamiltonian of the form

$$H=\frac{p^2}{2m}+V(x) \tag{34}$$

we find

$$\dot{x}=q\frac{p}{m} \qquad \dot{p}=-\frac{1}{q}V'(x) \tag{35}$$

and thus

$$m\ddot{x}=q\dot{p}=-V'(x) \tag{36}$$

so that the equation of motion for x remains unchanged. As a consistency check, we use the equations of motion to evaluate

$$\dot{H}=V'(x)p/m(q-1/q) \tag{37}$$

which is found to be in agreement with the equation

$$\dot{H}=\{H, H\}_q. \tag{38}$$

Furthermore, we note that although the Hamiltonian is not conserved, the equations of motion guarantee that the quantity

$$\tilde{H}=q\left(\frac{p^2}{2m}\right)+\frac{1}{q}V(x) \tag{39}$$

is conserved and, indeed, one readily sees that

$$\{\tilde{H}, H\}_q=0. \tag{40}$$

Thus the modification (33) is rather trivial, at least for Hamiltonians of the standard form (34). One learns, however, that the mere fact that the Hamiltonian is not conserved is not cause for undue alarm—the dynamics may nevertheless be perfectly sensible.

Next we return to the quantum case and study some simple examples to see what sort of dynamics arises from the q-deformed equations. In so doing, we shall, for simplicity, focus on the case in which there is one degree of freedom, $x(t)$, and its associated momentum, $p(t)$. Furthermore, we shall assume that x and p obey the usual canonical commutation relation $[x, p]=\mathrm{i}$. The reader may wonder whether it might not be more appropriate to use variables x', p' that obey the quommutation relation $[x', p']_q=\mathrm{i}$. There appears to be no principle, other than perhaps simplicity, to guide one's choice. Note, however, that these two alternatives are not necessarily all that different, in the following sense. Introduce the anti-Hermitian operator $G=(xp+px)/2\mathrm{i}$ with the properties

$$[G, p]=p \tag{41}$$

$$[G, x]=-x. \tag{42}$$

One then finds that with the definitions

$$x'=f(G)x \tag{43}$$

and

$$p' = h(G)p \tag{44}$$

(x', p') will obey the q-deformed Heisenberg algebra if (x, p) obey the undeformed algebra, provided

$$h(G+1) = \frac{1}{f(G)(G+\frac{1}{2})}\left(\frac{C}{q^{2G}} + \frac{q}{q^2-1}\right). \tag{45}$$

Here C is an arbitrary constant. Furthermore, if (x, p) are Hermitian, so will be (x', p'), provided that

$$q^2 = -C^*/C \tag{46}$$

and

$$f^*(G) = f[-(G+1)]. \tag{47}$$

(Here the notation is such that if $f(G) = \Sigma_n f_n G^n$ then $f^*(G) = \Sigma_n f_n^* G^n$.)

Equation (46) informs us that all this is possible only if $q = e^{i\theta}$, but that if this condition is met then there exists a whole family of transformations, labelled by functions $f(G)$ satisfying equation (47), that maps the undeformed canonical pair (x, p) into the deformed one†. This correspondence is only formal and deserves to be studied further, but for present purposes we shall assume that the pairs (x, p) and (x', p') are equivalent and therefore we shall stay with the former, which is simpler to calculate with.

One may also consider whether it may be possible to find mapping functions similar to $f(G)$ which transform H and any operator O so that they obey the usual commutator Heisenberg equations of motion rather than the quommutator equation (5). We have searched for such maps, but due to the right-hand side of equation (5) being another operator, and the time derivative of O itself to boot, rather than simply i as in the x, p case, our attempts to find such maps have been unsuccessful so far. To the extent that one may assume such maps do not exist, it seems that the quommutator deformation (5) is not trivially equivalent to the undeformed case.

Consider, then, the case of a 'free' particle:

$$H = \frac{p^2}{2m}. \tag{48}$$

p commutes with H and, hence, from our general discussion above,

$$p(t) = \frac{p_0}{1 - (irp_0^2/2m)t} \tag{49}$$

where $p_0 = p(0)$.

The behaviour of x is more complicated. The equation for its time evolution is

$$x(t) = \left(1 - \frac{ir}{2m}p_0^2 t\right)^{-q/r} x(0)\left(1 - \frac{ir}{2m}p_0^2 t\right)^{1/qr}. \tag{50}$$

One way to analyse this expression is to make formal use of the transform

$$A^p = \frac{1}{\Gamma(-p)}\int_0^\infty ds\, e^{-sA} s^{-(p+1)} \tag{51}$$

† The general solution to equation (47) has a Fourier transform of the form $\rho(k)\,e^{ik/2}$, where $\rho(k)$ is an arbitrary real function. We thank Charles Sommerfield for this remark.

to write

$$x(t)=\frac{1}{\Gamma(q/r)\Gamma(-1/qr)} \times \int_0^\infty \mathrm{d}s_1\,\mathrm{d}s_2\, s_1^{q/r-1} s_2^{-1/qr-1}\left(\mathrm{e}^{-s_1[1-(\mathrm{i}r/2m)p_0^2 t]}x(0)\,\mathrm{e}^{-s_2[1-(\mathrm{i}r/2m)p_0^2 t]}\right) \tag{52}$$

and then to use

$$\mathrm{e}^{(\mathrm{i}\lambda/2)p_0^2}x(0)\,\mathrm{e}^{-(\mathrm{i}\lambda/2)p_0^2}=x(0)+p_0\lambda \tag{53}$$

to obtain $x(t)$ in either of the equivalent forms

$$x(t)=\frac{1}{1-(\mathrm{i}r/2m)p_0^2 t}\,x(0)+\frac{1}{q}\frac{p_0 t}{[1-(\mathrm{i}r/2m)p_0^2 t]^2} \tag{54}$$

$$=x(0)\,\frac{1}{1-(\mathrm{i}r/2m)p_0^2 t}+q\frac{p_0 t}{[1-(\mathrm{i}r/2m)p_0^2 t]^2}. \tag{55}$$

As $q\to 1$ it is clear that one obtains the familiar time dependence associated with a free particle. Note that equations (54) and (55) exhibit $x(t)$ in two mutually Hermitian conjugate forms, so that in this case $x=x^\dagger$ for all t.

From equation (8), we expect that

$$H(t)=\frac{1}{2m}p^2(t)=\frac{H_0}{1-\mathrm{i}rH_0 t}.$$

One can check that this is true in the present example, provided that one recalls that, because time evolution is not unitary, there is a difference between $p^2(t)$ and $(p(t))^2$, and it is the former that one must use in evaluating $H(t)$.

As our second example, we choose the harmonic oscillator:

$$H=\tfrac{1}{2}(p^2+x^2) \tag{56}$$

(for convenience, we scale the mass and frequency of the oscillator to unity). Using the same transform technique, we may write

$$x(t)=\frac{1}{\Gamma(q/r)\Gamma(-1/qr)}\int_0^\infty \mathrm{d}s_1\,\mathrm{d}s_2\, s_1^{q/r-1}s_2^{-1/qr-1}\,\mathrm{e}^{-(s_1+s_2)}\tilde{x}(rs_1 t)\,\mathrm{e}^{\mathrm{i}rH_0(s_1+s_2)t} \tag{57}$$

$$=\frac{1}{\Gamma(q/r)\Gamma(-1/qr)}\int_0^\infty \mathrm{d}s_1\,\mathrm{d}s_2\, s_1^{q/r-1}s_2^{-1/qr-1}\,\mathrm{e}^{-(s_1+s_2)}\,\mathrm{e}^{\mathrm{i}rH_0(s_1+s_2)t}\tilde{x}(-rs_2 t) \tag{58}$$

with similar expressions for p. Here $\tilde{x}(t)$ denotes the usual quantum-mechanical time dependence of x:

$$\tilde{x}(t)=\mathrm{e}^{\mathrm{i}H_0 t}x(0)\,\mathrm{e}^{-\mathrm{i}H_0 t}. \tag{59}$$

Since for the harmonic oscillator case the functions $\tilde{x}(t)$ and $\tilde{p}(t)$ are particularly simple, one can perform the s_1 and s_2 integrals to obtain

$$x(t)=\tfrac{1}{2}\{(a-\mathrm{i}b)[1-\mathrm{i}r(H_0+1)t]^{-q/r}+(a+\mathrm{i}b)[1-\mathrm{i}r(H_0-1)t]^{-q/r}\}(1-\mathrm{i}rH_0 t)^{1/qr} \tag{60}$$

$$=\tfrac{1}{2}(1-\mathrm{i}rH_0 t)^{-q/r} \times\{[1-\mathrm{i}r(H_0+1)t]^{1/qr}(a+\mathrm{i}b)+[1-\mathrm{i}r(H_0-1)t]^{1/qr}(a-\mathrm{i}b)\} \tag{61}$$

and

$$p(t)=\tfrac{1}{2}\{(b+\mathrm{i}a)[1-\mathrm{i}r(H_0+1)t]^{-q/r}+(b-\mathrm{i}a)[1-\mathrm{i}r(H_0-1)t]^{-q/r}\}(1-\mathrm{i}rH_0t)^{1/qr} \quad (62)$$

$$=\tfrac{1}{2}(1-\mathrm{i}rH_0t)^{-q/r}$$
$$\times\{[1-\mathrm{i}r(H_0+1)t]^{1/qr}(b-\mathrm{i}a)+[1-\mathrm{i}r(H_0-1)t]^{1/qr}(b+\mathrm{i}a)\} \quad (63)$$

where $a=x(0)$ and $b=p(0)$.

These results could also have been obtained, perhaps more quickly, by observing that $a\pm\mathrm{i}b$ are the raising and lowering operators for H_0, but the expressions (57) and (58) have the advantage of greater generality.

We now consider another deformation of the Heisenberg equations of motion, motivated by the deforming part of the definition of $\mathrm{su}(2)_q$ given by equation (2*b*). Thus, we assume the equation of motion of an operator $X(t)$ to be given by

$$\mathrm{i}[H(t),X(t)]=[\dot{X}(t)]_q. \quad (64)$$

Again, as $q\to 1$ we recover the usual equations of motion. Unlike the quommutator case of equation (5), this deformation has the obvious advantage that, as usual, operators commute with themselves, so that, in particular, the Hamiltonian itself is a constant of the motion and energy is conserved. Furthermore, Hermiticity appears to be preserved in the cases when q is treated like a formal variable or when $q=\mathrm{e}^{\mathrm{i}\theta}$.

In order formally to integrate equation (64), let us write $q=\mathrm{e}^h$. Then $[x]_q=\sinh(hx/2)/\sinh(h/2)$, and, writing $\tilde{h}=\sinh(h/2)$, we find, for $X(t)$,

$$\frac{2t}{h}=\int_{X(0)}^{X(t)}\frac{\mathrm{d}X}{\sinh^{-1}(\mathrm{i}\tilde{h}[H,X])}. \quad (65)$$

Because of the highly non-trivial nonlinearities, it is difficult to go further in the general case. Thus we cannot make the type of general statements about the transformation to, and the results in, the Schrödinger picture which we were able to make in the quommutator case. Equation (64) can probably be represented in the Schrödinger representation, if at all, only by some version of a nonlinear Schrödinger equation. (Recent examples of such equations are discussed in [9].)

However, for the free particle the differential equations separate simply, and p is constant, so that we find, for $x(t)$,

$$x(t)-x(0)=\frac{2t}{h}\sinh^{-1}(\tilde{h}p) \quad (66)$$

which is a well-behaved, single-valued function. This result is certainly different from the usual quantum-mechanical result, $x(t)=pt$, but not only is it quite sensible, it also reduces to this usual behaviour as $q\to 1$. It is also apparent that the behaviour given by equation (66) is very different from that obtained from the quommutator deformation, equations (54) and (55). Time evolution is certainly unitary here in this free case. Although we have not yet been able to establish unitarity in general, we suspect it to be there. These are yet more indications that the two methods are not equivalent.

When we seek to go beyond the free particle, the nonlinearity of this deformed quantum mechanics confronts us squarely. Even for the harmonic oscillator, the coupled nonlinear differential equations

$$\dot{x}(t)=\left(\frac{2}{h}\right)\sinh^{-1}(\tilde{h}p(t)) \qquad \dot{p}(t)=\frac{2}{h}\sinh^{-1}(\tilde{h}x(t)) \quad (67)$$

do not lend themselves to a closed-form solution, although a numerical solution is, of course, possible.

3. Discussion

The two methods we have studied are not by any means the only possible deformations that one could be inspired to consider by studying the literature on quantum groups. For example, there exists the so-called quantum derivative [10]

$$D_u f(u) \equiv \frac{f(q^{1/2}u) - f(q^{-1/2}u)}{u(q^{1/2} - q^{-1/2})} \tag{68}$$

which goes over into the usual derivative as $q \to 1$. This arises, for example, when one notices that it is possible to deform the following representation of SU(2) acting on functions of two variables u and v,

$$J_+ = u\frac{\partial}{\partial v} \qquad J_- = v\frac{\partial}{\partial u} \qquad J_0 = \left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)$$

by letting $J_+ = uD_v$ and $J_- = vD_u$ (while letting J_0 alone), thereby satisfying the quantum algebra (2). This suggests at least two other deformations of the Heisenberg equations:

$$(a) \qquad D_t X = \mathrm{i}[H, X] \tag{69}$$

and

$$(b) \qquad D_t X = \mathrm{i}[H, X]_q . \tag{70}$$

Choice (a) is somewhat similar to our second method (equation (64)) and has the same advantage that H should be a constant, unlike equation (5) or choice (b). However, unlike equation (5), which leads to the integrated form (equation (9)), choices (a) or (b) are even harder to integrate than equation (64). So on purely pragmatic grounds we have concentrated mostly on equation (5).

The quantum deformation (5) has some manifestly unpleasant properties, chief among them that: (i) H is not a constant; (ii) time evolution is not unitary; (iii) the norm of a state can become infinite and, in general, tends to vanish for large times. Of these, the first is perhaps the least worrisome since, as we saw in the classical case, the dynamics may nevertheless be perfectly sensible, and also, even though $\dot{H} \neq 0$, there are plenty of constant operators about, such as H_0, whose eigenstates can be used as a basis and whose eigenvalues can be thought of as energy.

Properties (ii) and (iii) are clearly related, since if time evolution were unitary, the norm of a state could not change. What the lack of unitarity seems to imply is that, just as in the passage from classical to ordinary quantum mechanics one gives up determinism in favour of probability amplitudes that are defined as matrix elements in Hilbert space, here one is sacrificing the physical significance of the Hilbert space inner product but, it appears, retaining the notion of an expectation value, which is an appropriate ratio of such inner products.

If anything has been gained from such a sacrifice it may be the suggestion that the quantum deformation parameter acts as a cut-off that discretizes the time evolution of the system. This suggestion arises in at least two ways. First, the quommutator in the context of quantum groups is equivalent to the introduction of the quantum derivative, which is an obvious discretization of the ordinary derivative. Second, inspection of equation (9) reveals that the evolution operator $S(t)$ resembles a discretization of the exponential $\mathrm{e}^{-\mathrm{i}H_0 t}$. In fact, if we write

$$\mathrm{e}^{-\mathrm{i}H_0 t} = \lim_{n\to\infty}\left(1 - \mathrm{i}H_0\frac{t}{n}\right)^n \tag{71}$$

we see that $S(t)$ can be thought of as the evolution operator appropriate to $n = 1/qr$ discrete steps, with Hamiltonian $H_{\rm discr} = (1/q)H_0$ (cf equation (12)), provided we agree to overlook the fact that $n = 1/(e^{2i\theta} - 1)$ is not exactly a positive integer.

A small technical point is that neither the quommutator nor the deformed Poisson bracket of equation (32) obeys the usual Jacobi identity. Of course they each obey a suitably deformed version of the Jacobi identity. (This follows immediately because the Jacobi identity holds when $q = 1$.) The fact that our Poisson bracket is non-associative explains why it is not a special case of the Moyal bracket, which, as Fletcher has shown [11], is the most general associative deformation of the Poisson bracket.

The examples we have studied explicitly involved only a single degree of freedom. In order to make these ideas relevant to, say (to choose a wild example), the regularization of quantum gravity, one must understand how to extend them to a field-theoretic setting. This extension has yet to be investigated.

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